arXiv:cond-mat/0411228v2 [cond-mat.other] 22 Mar 2005

Phase-matched four wave mixing and quantum beam splitting of matter waves in a periodic potential

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We show that the dispersion properties imposed by an external periodic potential ensure both energy and quasi-momentum conservation such that correlated pairs of atoms can be generated by four wave mixing from a Bose-Einstein condensate moving in an optical lattice potential. In our numerical solution of the Gross-Pitaevskii equation, a condensate with initial quasi-momentum k_0 is transferred almost completely ($> 95\%$) into a pair of correlated atomic components with quasi-momenta k_1 and k_2, if the system is seeded with a smaller number of atoms with the appropriate quasi-momentum k_1.

PACS numbers: 03.75.Kk, 03.75.Lm, 05.45.-a

Bose-Einstein condensates in optical lattices provide flexible systems for studying the behavior of coherent matter in periodic potentials. Considerable attention is given to studies in regimes far from the region of validity of mean-field analysis and the Gross-Pitaevskii equation [1], but also the highly non-trivial mean-field dynamics has been and continues to be subject to theoretical and experimental investigations [2, 3, 4].

The process we wish to consider is a four wave mixing (FWM) process, which transfers pairs of atoms coherently from an initial momentum state k_0 to new states with momenta k_1 and k_2. We consider a Bose-Einstein condensate in a quasi-1D geometry and we will consider only the longitudinal dynamics of the condensate. This geometry is relevant, e.g., for atomic wave guides and atom interferometers based on atom chips [5].

In Refs. [6, 7], it was shown that nonlinear interaction originating from the s-wave scattering between atoms leads to depletion of the condensate and emission of pairs of atoms at other momenta when a continuous matter wave beam passes through a finite region with enhanced interactions. For a larger condensate, however, the process will not be effective unless it conserves both energy and momentum, i.e., the waves must be phase-matched over the extent of the sample. We shall show how the characteristic energy band structure in a one-dimensional optical lattice can be used to ensure both energy and quasi-momentum conservation, i.e., phase-matching of the FWM process.

Our tailoring of the dispersion properties of matter waves by an external potential is inspired by approaches to non-linear optics, which employ various means to ensure phase-matching, e.g., of the FWM process [8, 9, 10]. A similar phase-matched FWM process has been used to explain giant amplification from semiconductor microcavities, where the polariton dispersion properties can be controlled by the strong photon-exciton coupling [11]. We also note that a recent analysis [12] of the break-up of a bright matter wave soliton was analyzed in terms of dispersion and phase-matching. Phase-matched FWM has been realized in collisions of two condensates in two dimensions [13, 14, 15], but in the present paper we show that the process can take place with atomic motion along a single direction, for example inside an atomic waveguide.

The basic idea of our proposal is illustrated in Fig. 1. In a periodic potential $V(z)$, the energy spectrum constitutes a band structure, and the figure shows the lowest energy band for the corresponding linear Schrödinger equation. When two atoms with momentum k_0 collide and leave with momenta k_1 and k_2 momentum conservation requires

$$2k_0 = k_1 + k_2 \ modulo\ Q, \tag{1}$$

where Q is a reciprocal lattice vector. In the periodic potential the energy does not vary quadratically with the wave number, and as indicated by example in Fig. 1, it is possible to identify sets of wave numbers with conservation of the total energy

$$2\varepsilon_0 = \varepsilon_1 + \varepsilon_2. \tag{2}$$

To investigate the effectiveness of this degenerate FWM process, we have performed a mean-field analysis of the dynamics of the condensate based on the one-dimensional Gross-Pitaevskii equation

$$i\hbar\frac{d\Psi}{dt} = \Big(- \frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V(z) + \gamma|\Psi|^2\Big)\Psi, \tag{3}$$

where the periodic potential $V(z)$ is given by

$$V(z) = -\beta E_R cos(2k_L z), \tag{4}$$

where $E_R \equiv \hbar^2 k_L^2/2m$ is the recoil energy. The periodic potential can be generated with a standing wave of a laser with wavelength $\lambda = 2\pi/k_L$. The factor

$$\gamma = gN/A_\perp \tag{5}$$

describes the strength of the nonlinearity, where N is the total number of atoms in the condensate, $A_\perp$ is the area

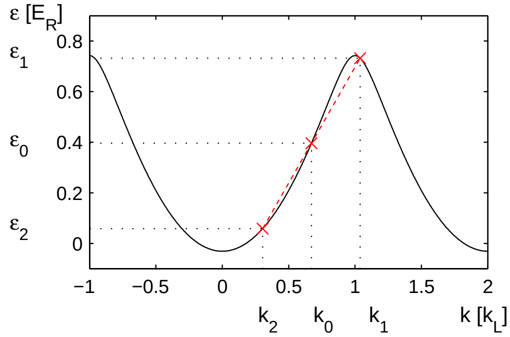

FIG. 1: Band structure for atomic motion in the periodic potential Eq. (4). Quasi-momentum conservation and energy conservation is fulfilled in the crosses where two atoms with momentum k_0 collide and separate at momenta k_1 and k_2 illustrated in the figure.

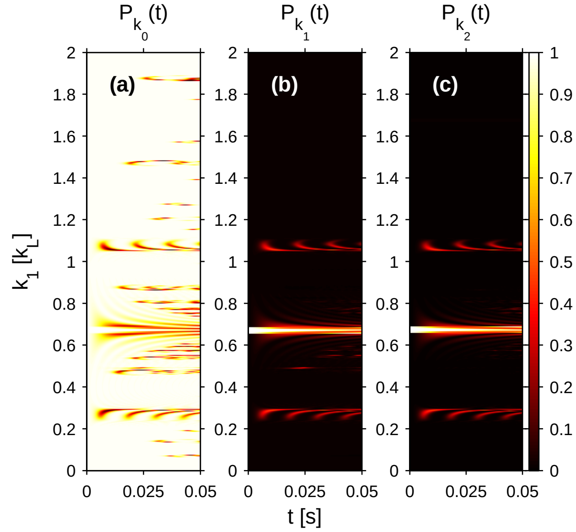

FIG. 2: Population of different momentum components (a): $P_{k_0}(t)$, (b): $P_{k_1}(t)$ and (c): $P_{k_2}(t)$ as a function of time and as function of the seeding wave vector k_1. The calculations are performed with a potential modulation $\beta = 1/2$, $N = 100000$ atoms, and an initial wave vector of $k_0 = 0.672k_L$. When (k_0, k_1, k_2) fulfill the phase-matching conditions in Eq. (1) and (2), which is the case for $k_1 = 0.289k_L,\ 1.055k_L$, the condensate originally having wave vector k_0 is efficiently transferred into a set of atomic clouds with wave vectors k_1 and k_2.

of the transverse ground state, and g relates to the s-wave scattering length a_s and the mass m of the atoms as $g = 4\pi\hbar^2 a_s/m$. In our calculation presented below we assume $\beta = 1/2$ and $\gamma = 40.8E_R/k_L$ corresponding to $N = 100000\ ^{87}Rb$ atoms confined to a transverse area of $A_\perp = 42\mu\mathrm{m}^2$ and a grid of 512 periods of the potential. The effective area $A_\perp = \frac{2\pi\hbar}{m\omega_\perp}\sqrt{1 + 2a_sN/L}$ results from a gaussian variational ansatz [16, 17], to the radial distribution in a harmonic potential with $\omega_\perp = 2\pi \times 44$ Hz and a constant longitudinal density N/L.

An eigenstate Ψ_0 of the condensate with quasi-momentum k_0 is found using the method of steepest descent in imaginary time, assuming a solution according to Bloch's theorem on a single period of the lattice potential. Subsequently, a seed at variable k_1 is applied giving the following initial wave function

$$\Psi_{init}(z) = \frac{1}{\sqrt{1+\alpha^2}}\left[1 + \alpha e^{i(k_1-k_0)z}\right]\Psi_0(z), \qquad (6)$$

where $\alpha = 0.1$ has been used in our calculations with the Gross-Pitaevskii equation. This wave function does not fulfill Bloch's theorem, and we hence restrict ourselves to values of k_0 and k_1 with interference patterns which are periodic on an extended grid of 512 periods of the potential. To test the importance of phase-matching in the FWM process we propagate the wave function Ψ_{init} on this grid, with different values for the seeded momentum component k_1. As a function of time, we can observe the evolution of the Gross-Pitaevskii wave function and build-up of amplitude at different momenta. We are particularly interested in the quasi-momentum regions around k_0, k_1 and $k_2 = 2k_0 - k_1$. Let $\psi(k,t)$ denote the Fourier transform of the time-dependent Gross-Pitaevskii wave-function $\Psi(z,t)$. The distribution in momentum space folded into the single Brillouin zone from $k = 0$ to $Q = 2k_L$ is given by the following expression

$$P_k(t) = \sum_n \int_{k-\Delta k/2}^{k+\Delta k/2} |\psi(k+nQ,t)|^2 dk, \qquad (7)$$

where we have introduced a sampling over a narrow momentum window with $\Delta k = k_L/32$.

Fig. 2(a) shows the part of the condensate, $P_{k_0}(t)$, remaining at the initial quasi-momentum k_0 when the condensate is seeded with different values of k_1. The most important features in the figure occur when the condensate is seeded with $k_1 = 1.055k_L$ and $k_1 = 0.289k_L$. The original condensate fraction at k_0 is almost completely depleted, and strong growth of the population of the seeded momentum state is shown in $P_{k_1}(t)$ in part b, accompanied by simultaneous growth in the phase-matched component $k_2 = 2k_0 - k_1$, shown as $P_{k_2}(t)$ in part c of the figure. Comparing the set $(k_0, k_1, k_2) = (0.672k_L, 1.055k_L, 0.289k_L)$ with the set of phase-matched wave vectors in Fig. 1, we find extremely good agreement and we conclude that the structure at $k_1 = 1.055k_L$ is indeed due to the phase-matched FWM process.

The remaining structures in Fig. 2 are less prominent but for instance the structure in Fig. 2(b) after 25 msec at $k_1' = 0.492k_L$ can be identified as a double FWM process with the following steps: $2k_0 \to k_1' + k_2'$ and $k_0 + k_2' \to k_1' + k_3'$, where $k_3' = k_2' + (k_0 - k_1')$. These steps do not

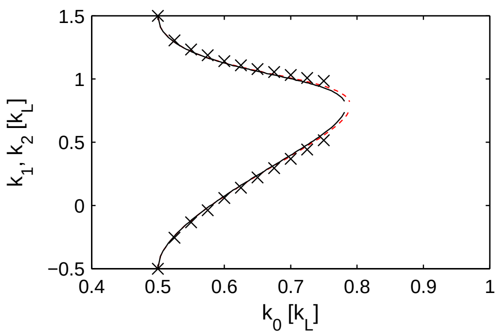

FIG. 3: Identification of momentum states (k_0, k_1, k_2) fulfilling the phase-matching condition. Crosses: sets of phase-matched values (k_0, k_1, k_2) identified by numerical solution of the Gross-Pitaevskii equation (3). Full line: phase-matching condition derived from band structure based on simple linear Schrödinger equation. Dashed line: phase-matching condition when the effects of interactions have been included as in Eq. (8). Although the simple band structure calculations are not exact theories for the phase-matching condition there is still excellent agreement with the full mean-field calculation.

conserve energy but the resulting six wave mixing process $(3k_0 \to 2k_1' + k_3')$ is resonant. A closer examination indeed confirms that the k_1' component is twice as populated as the k_3' population in the region of the bright spot on the figure. It is again the phase-matching that is responsible for the significance of this amazing process.

We have performed calculations with the Gross-Pitaevskii equation (3) for various k_0 and identified the phase-matched sets of wave vectors (k_0, k_1, k_2), plotted as crosses in Fig. 3. To quantitatively understand the occurring sets of wave vectors we performed a simple band structure calculation based on a linear Schrödinger equation and extracted the sets of wave vectors fulfilling energy and quasi-momentum conservation as illustrated in Fig. 1. The phase-matched quasi-momenta k_0 occur within the interval: $k_L/2 < k_0 < k_{zm}$, where $k_L = Q/2$ and k_{zm} is a point of zero effective mass in the band structure $(\frac{d^2\varepsilon}{dk^2}\big|_{k_{zm}} = 0,\ 0 < k_{zm} < k_L)$. Those sets of wave vectors are plotted as the full line in Fig. 3. This simple procedure produces results very much in agreement with those obtained by the full numerical solution. For completeness we present also a calculation including the effect of interactions. To calculate the band structure for quasi-momentum states of atoms outside the k_0 condensate state, we assume that these are particle-like excitations, and we solve the following equation:

$$\Big(-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + V(z) + 2\gamma|\Psi_0(z)|^2 \Big) u(z) = \varepsilon u(z), \quad (8)$$

sometimes referred to as the Popov approximation to the coupled Bogoliubov-de Gennes equations for the problem. The mean-field interaction term $2\gamma|\Psi_0(z)|^2$ depends on k_0 and amounts to less than a five percent correction to $V(z)$. Hence, we identify $\Psi_0(z)$ and we solve Eq. (8) for each quasi-momentum k_0 and derive band structures, from which we extract the phase-matched pair k_1, k_2 of final momenta. Sets of (k_0, k_1, k_2) found with this method are shown by the dashed curve in Fig. 3. As expected the interactions only slightly change the phase-matching condition.

Our original expectations were that the phase-matched, degenerate FWM could be achieved in a perturbative regime with only few atoms expelled from the original condensate. When the process is phase-matched, however, the calculations show an extremely high conversion efficiency (up to 95%). Furthermore the populations $P_{k_i}(t)$, displayed more clearly in Fig. 4(a)-(c) for the set of phase-matched wave vectors $(k_0, k_1, k_2) = (0.672k_L, 1.055k_L, 0.289k_L)$, show clear oscillatory behavior. Such Rabi-like oscillations are normally met in transitions between discrete states, but, as illustrated in Fig. 1, increasing k_1 (and decreasing k_2) lowers the energy of both states, whereby energy conservation restricts the coupling to a narrow part of the momentum state continuum, in which case the dynamics passes to Rabi oscillatory dynamics [18]. The expected frequency of the Rabi-oscillations is proportional to N. Fig. 4(d)-(f) illustrate $P_{k_0}(t), P_{k_1}(t), P_{k_2}(t)$ for the phase-matched $(k_0, k_1, k_2) = (0.672k_L, 1.047k_L, 0.297k_L)$ from a Gross-Pitaevskii simulation with half the amount of atoms as compared with Fig. 4(a),(b),(c), and, indeed, we observe approximately half the Rabi-oscillation frequency. Note the slightly different values of (k_1, k_2) due to the effect of the interactions on the phase-matching condition.

Assuming the rather strict final state selectivity due to energy and momentum conservation we have performed a number state analysis in a few mode basis based on the three populated quasi-momentum states. We have thus written the wave function as $\Psi = \sum_{n_{k_0},n_{k_1},n_{k_2}} c_{n_{k_0},n_{k_1},n_{k_2}} |n_{k_0}, n_{k_1}, n_{k_2}> = \sum_n c_n |N-s-2n, s+n, n>$ and calculated the evolution of the c_n coefficients, starting with $N-s$ atoms in k_0 and s in k_1, and evolving under a Hamiltonian that removes pairs of k_0 atoms and creates k_1, k_2 pairs. This calculation confirms the oscillatory behavior between the mean number of atoms in the k_0 state and in the k_1, k_2 states. While the numerical solution of the Gross-Pitaevskii equation shows that even a tiny seed leads to the same conversion, but delayed with respect to the results of a larger seed, our number state calculation shows that it is necessary to seed the condensate by a finite amount to avoid the decoherence due to strongly n-dependent coupling amplitudes. With sufficient seed our number state analysis reproduces the macroscopic conversion and population oscillations observed in the mean field analysis.

In conclusion the process of phase-matched FWM in a BEC in a periodic potential can be extremely efficient. When seeded, up to 95% of the atoms originally in the condensate with wave vector k_0 can be transferred into

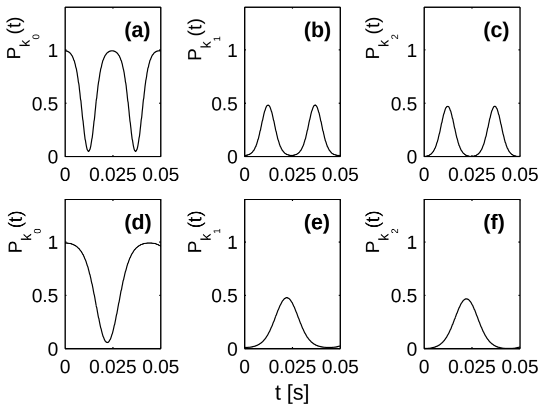

FIG. 4: The condensate evolution based on the Gross Pitaevskii equation for sets of phase-matched wave vectors: (a): $P_{k_0=0.672k_L}(t)$, (b): $P_{k_1=1.055k_L}(t)$ and (c): $P_{k_2=0.289k_L}(t)$ for $\beta = 1/2$ and $N = 100000$. The evolution of the phase-matched components of the condensate with half the number of atoms ($N = 50000$) is shown in (d): $P_{k_0=0.672k_L}(t)$, (e): $P_{k_1=1.047k_L}(t)$ and (f): $P_{k_2=0.297k_L}(t)$. Half the oscillation frequency is observed since the oscillation frequency is proportional to the number of atoms.

the correlated pair of states with k_1 and k_2. The system can be used as a source of correlated atomic clouds where pairs can be easily separated due to the rather large momentum difference between k_1 and k_2. Because of the coupling to a very narrow continuum of states the system performs Rabi-oscillations between the k_0 and the k_1, k_2 states, a behavior confirmed by the number state analysis with few modes.

Potentially better descriptions than our simple Gross-Pitaevskii equation [17] have been proposed to describe transversely confined elongated condensates. Our main mechanism relies on the non-trivial band structure, but not on its particular shape, and we therefore believe that our proposal should remain generally valid. Transverse confinement is, however, an important issue, and it is a natural extension of our theory to consider transverse excitations and, more generally, motion in 2D or 3D lattices, where energy conservation and phase-matching may lead to a range of interesting solutions.

Theories [2, 17] have proposed and experiments [19, 20] have shown that condensates moving in periodic potentials become unstable for certain ranges of quasi-momenta. These results are linked with the energy and momentum conserving processes identified in this paper, but they also involve the detailed properties of the transverse confinement [17]. Our calculations assume a lower density of atoms than in the experiments reported in [19], and in this regime we expect that only few atom pairs will be spontaneously scattered, and that our seeded process will be dominant.

K.M. Hilligsøe acknowledges NKT Academy for financial support and Kevin Donovan for carefully reading the manuscript.

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[1] I. Bloch, Physics World **April**, 25 (2004).
[2] B. Wu and Q. Niu, New J. Phys. **5**, 104 (2003).
[3] O. Morsch and E. Arimondo, in "Lecture Notes in Physics", vol. 602, T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens Eds., Springer, 312 (2002), arXiv:cond-mat/0209034 (2002).
[4] K. M. Hilligsøe, M. K. Oberthaler, and K.-P. Marzlin, Phys. Rev. A **66**, 063605 (2002).
[5] P. Hommelhoff, W. Hänsel, T. Steinmetz, T. W. Hänsch, and J. Reichel, submitted to New Journal of Physics, arXiv:quant-ph/0411012 (2004).
[6] W. Zhang, C. P. Search, H. Pu, P. Meystre, and E. M. Wright, Phys. Rev. Lett. **90**, 140401 (2003).
[7] I. Bouchoule and K. Mølmer, Phys. Rev. A **67**, 011603(R) (2003).
[8] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, 2001), 3rd ed.
[9] K. M. Hilligsøe, T. V. Andersen, H. N. Paulsen, C. K. Nielsen, K. Mølmer, S. Keiding, R. Kristiansen, K. P. Hansen, and J. J. Larsen, Opt. Express **12**, 1045 (2004).
[10] T. V. Andersen, K. M. Hilligsøe, C. K. Nielsen, J. Thøgersen, K. P. Hansen, S. R. Keiding, and J. J. Larsen, Opt. Express **12**, 4113 (2004).
[11] P. G. Savvidis, J. J. Baumberg, R. M. Stevenson, M. S. Skolnick, D. M. Whittaker, and J. S. Roberts, Phys. Rev. Lett **84**, 1547 (2000).
[12] A. V. Yulin, D. V. Skryabin, and P. S. J. Russell, Phys. Rev. Lett. **91**, 260402 (2003).
[13] J. M. Vogels, K. Xu, and W. Ketterle, Phys. Rev. Lett. **89**, 020401 (2002).
[14] L. Deng, E. W. Hagley, J. Wen, M. Trippenbach, Y. Band, P. S. Julienne, J. E. Simsarian, K. Helmerson, S. L. Rolston, and W. D. Phillips, Nature **398**, 218 (1999).
[15] M. Trippenbach, Y. B. Band, and P. S. Julienne, Opt. Express **3**, 530 (1998).
[16] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A **65**, 043614 (2002).
[17] M. Modugno, C. Tozzo, and F. Dalfovo, arXiv:cond-mat/0405653 (2004).
[18] C. Cohen-Tannoudjii, J. Dupont-Roc, and G. Grynberg, *Atom-photon interactions. Basic processes and applications* (John Wiley and Sons, Inc., New York, 1992).
[19] S. Burger, F. S. Cataliotti, C. Fort, F. Minardi, M. Inguscio, M. L. Chiofalo, and M. P. Tosi, Phys. Rev. Lett. **86**, 4447 (2001).
[20] L. Fallani, L. D. Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Phys. Rev. Lett. **93**, 140406 (2004).